

Dilatonic BTZ black holes with power-law field

S. H. Hendi^{1,2*}, B. Eslam Panah^{1,2 †}, S. Panahiyan^{1,3 ‡} and A. Sheykhi^{1,2§}

¹Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran

²Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), Maragha, Iran

³Physics Department, Shahid Beheshti University, Tehran 19839, Iran

Motivated by low energy effective action of string theory and large applications of BTZ black holes, we will consider minimal coupling between dilaton and nonlinear electromagnetic fields in three dimensions. The main goal is studying thermodynamical structure of black holes in this set up. Temperature and heat capacity of these black holes are investigated and a picture regarding their phase transitions is given. In addition, the role and importance of studying the mass of black holes is highlighted. We will see how different parameters modify thermodynamical quantities, hence thermodynamical structure of these black holes. In addition, geometrical thermodynamics is used to investigate thermodynamical properties of these black holes. In this regard, the successful method is presented and the nature of interaction around bound and phase transition points is studied.

I. INTRODUCTION

Appearing a scalar field in the low energy limit of string theory has motivated various scientists to study dilaton gravity with different viewpoints. The coupling of dilaton with other gauge fields has a profound effects on the resulting solutions [1–6]. For example, it was shown that the dilaton field can change the asymptotic behavior of the spacetime. In particular, it was argued that in the presence of one or two Liouville-type dilaton potentials, black hole spacetimes are neither asymptotically flat nor (anti)-de Sitter ((A)dS) (see [7–15] and references therein). This is due to the fact that the dilaton field does not vanish for $r \rightarrow \infty$. Latter, it was shown that with combination of three Liouville type dilaton potentials, it is possible to construct dilatonic black hole solutions in the background of (A)dS spacetime [16–20]. Recently, studies on compact objects such as neutron stars in the context of dilaton gravity [21, 22] as well as black objects in dilaton gravity's rainbow have been carried out [23, 24].

On the other side, one of the important solutions of Einstein's field equation is three dimensional black holes. The first three dimensional black hole solution was found by Banados-Teitelboim-Zanelli (BTZ) [25], and afterwards, BTZ black holes have got a lot of attentions [26–47]. The motivation to study three dimensional solutions originates from the fact that near horizon geometry of these solutions serves as a worthwhile model to investigate some conceptual questions of AdS/CFT correspondence [48, 49]. Also, studying the BTZ black hole has improved our knowledge of gravitational systems and their interactions in 3-dimensions [48]. Furthermore, possible existence of gravitational Aharonov-Bohm effect due to the noncommutative BTZ black holes [50], and specific relations between these black holes and effective action in string theory have been explored [51–53]. From the viewpoint of quantum theory of gravity, three dimensional gravity plays an important role. Entanglement, quantum entropy [54–56], holography and holographical superconductors of the BTZ black holes have been studied in Refs. [57–62].

The studies on the BTZ black holes/wormholes were also extended to include the dilaton field [63, 64], the nonlinear electrodynamics [65–68], and higher dimensional spacetimes [69, 70]. Also, BTZ black holes in the presence of massive gravity with Maxwell and Born-Infeld fields have been studied in Ref. [71]. In particular, it was shown that the electric field of BTZ-like solutions in higher-dimensions are the same as three dimensions and they are thermally stable in the whole phase space [69, 70]. In [72], thermodynamical stability of static uncharged BTZ dilaton black holes in the canonical ensemble was explored. It was shown that depending on the dilaton coupling constant, α , the solution can exhibit a stable phase. Indeed, it was observed that the system is thermally stable/unstable for specific values of the dilaton parameter.

On the other hand, gravity coupled with the nonlinear electrodynamics attracts significant attentions because of its specific properties in gauge/gravity coupling. Interesting physical properties of various nonlinear electrodynamic models have been studied in many papers [67–71, 73–83]. One of the interesting branches of the nonlinear electrodynamic models is related to power Maxwell invariant (PMI) theory in which its Lagrangian is an arbitrary power of Maxwell Lagrangian [84–86]. The PMI theory has more interesting properties with regard to the Maxwell field, and for unit power it reduces to linear case (Maxwell theory). One of the attractive properties of this theory is related to

* email address: hendi@shirazu.ac.ir

† email address: behzad.eslampanah@gmail.com

‡ email address: sh.panahiyan@gmail.com

§ email address: asheykhi@shirazu.ac.ir

the possible conformal invariance. It is notable that, when the power of Maxwell invariant is a quarter of spacetime dimensions ($power = dimensions/4$), this theory enjoys the conformal invariance. In other words, by considering power of Maxwell invariant equal to $dimensions/4$, one can obtain a traceless energy-momentum tensor which leads to an invariant theory under conformal transformation. It is notable that, considering this conformal symmetry, we can construct charged black hole with an inverse square electric field in arbitrary dimensions which is analogue to the four-dimensional Reissner-Nordeström solutions [87].

It is worth mentioning that the BTZ black holes in dilaton and dilaton gravity's rainbow have been investigated in Refs. [24, 72]. Thus it is well motivated to consider the three dimensional dilatonic black holes in the presence of different gauge fields. Since exact solutions of charged dilatonic black holes have not been investigate before, in the present work, one of our goals is obtaining an exact solution of three dimensional dilatonic black holes by adding the PMI electrodynamics to the action. In addition, we try to study interesting properties of such exact solution as a thermodynamical system.

One of the interesting methods for studying properties of thermodynamical systems is through geometrical thermodynamics. This method employs Riemannian geometry to construct phase space. The information regarding thermodynamical behavior of a system could be extracted from the Ricci scalar of this phase space. Divergencies of the Ricci scalar determine two sets of important points: bound points which separate situations with positive temperature from those with negative temperature, and phase transition points which represent discontinuities in thermodynamical quantities such as the heat capacity. In addition, considering the sign of Ricci scalar around its divergencies, it is possible to determine whether the interaction around divergencies are of repulsive or attractive nature [88].

The first geometrical thermodynamical approach was introduced by Weinhold which is based on internal energy as thermodynamical potential [89, 90]. Latter, an alternative was proposed by Ruppeiner which has entropy as its thermodynamical potential [91, 92]. Another proposal for the geometrical thermodynamics was given by Quevedo which has Legendre invariance as its core stone [93, 94]. Mentioned methods have been used in studying thermodynamics of the black holes [95–103]. In a series of papers, it was shown that the mentioned methods may confront specific problems in describing thermodynamical properties of the black holes [104–107]. In other words, there were cases in which, the results of these three approaches were not consistent with those extracted from other methods. In order to remove the shortcomings of other methods regarding geometrical thermodynamics, a new thermodynamical metric was introduced by Hendi, Panahiyan, Eslam Panah and Momennia (HPEM) [104]. It was shown that employing this method leads to consistent results regarding thermodynamical properties of the black holes. For a comparative study regarding these four thermodynamical metrics, we refer the readers to Ref. [108].

The organization of our paper is as follows. In the next section, we present the field equations of three dimensional charged dilatonic black holes when the gauge field is in the form of power Maxwell field. In section III, we calculate conserved and thermodynamic quantities of obtained solutions and examine the validity of the first law of thermodynamics. Thermodynamical behavior of the solutions and the properties governing it are explored in different contexts in sections IV and V. The paper is concluded in section VI with some closing remarks.

II. FIELD EQUATION AND CHARGED DILATON BLACK HOLE SOLUTIONS

In this section, we obtain three dimensional charged dilatonic black hole solutions in the presence of PMI field. For this purpose, we consider a 3–dimensional action of Einstein gravity which is coupled with dilaton and PMI fields

$$\mathcal{I} = \frac{1}{16\pi} \int d^3x \sqrt{-g} \left[\mathcal{R} - 4(\nabla\Phi)^2 - V(\Phi) + (-e^{-4\alpha\Phi}\mathcal{F})^s \right], \quad (1)$$

where \mathcal{R} is the Ricci scalar, Φ is the dilaton field and $V(\Phi)$ is a dilatonic potential. Also, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant, in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor with the electromagnetic potential A_μ and s the power of nonlinearity. In addition, it should be pointed out that α is a constant which determines the strength of coupling of the scalar and electromagnetic field.

Using variational principle and varying Eq. (1) with respect to the gravitational, dilaton and gauge fields ($g_{\mu\nu}$, Φ and A_μ), we find the following field equations

$$R_{\mu\nu} = 4 \left(\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right) + e^{-4\alpha\Phi} (-\mathcal{F})^s \left[(2s-1) g_{\mu\nu} - \frac{2s}{\mathcal{F}} F_{\mu\lambda} F_\nu{}^\lambda \right], \quad (2)$$

$$\nabla^2 \Phi = \frac{1}{8} \frac{\partial V}{\partial \Phi} + \frac{s\alpha}{2} e^{-4s\alpha\Phi} (-\mathcal{F})^s, \quad (3)$$

$$\partial_\mu \left(\sqrt{-g} e^{-4\alpha s \Phi} (-\mathcal{F})^{s-1} F^{\mu\nu} \right) = 0. \quad (4)$$

In this paper, we are attempting to obtain charged dilatonic black hole solutions in $(2+1)$ -dimensions. To do so, one can employ following static metric ansatz

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) d\varphi^2, \quad (5)$$

where $f(r)$ and $R(r)$ are radial dependent functions which should be determined.

Since we are looking for the black holes with a radial electric field ($F_{tr}(r) = -F_{rt}(r) \neq 0$), the electromagnetic potential will be in the following form

$$A_\mu = \delta_\mu^0 h(r). \quad (6)$$

Using Eqs. (4) and (6), and considering the metric (5), one can obtain electromagnetic tensor as

$$F_{tr} = E(r) = q e^{\frac{4\alpha s \Phi}{2s-1}} [rR(r)]^{\frac{1}{1-2s}}, \quad (7)$$

where q is an integration constant which is related to the electric charge of black holes.

Now, we can obtain field equations by using Eqs. (2) and (3) as

$$eq_{tt} : 2^s (s-1) \left(E^2(r) e^{-4\alpha\Phi(r)} \right)^s + \frac{1}{2} \left[f''(r) + \left(\frac{1}{r} + \frac{R'(r)}{R(r)} \right) f'(r) \right] + V(\Phi) = 0, \quad (8)$$

$$eq_{rr} : eq_{tt} + \left[\frac{R''(r)}{R(r)} + \frac{2R'(r)}{rR(r)} + 4\Phi'^2(r) \right] f(r) = 0, \quad (9)$$

$$eq_{\theta\theta} : 2^s (1-2s) \left(E^2(r) e^{-4\alpha\Phi(r)} \right)^s + \left[\frac{R''(r)}{R(r)} + \frac{2R'(r)}{rR(r)} \right] f(r) + \left[\frac{1}{r} + \frac{R'(r)}{R(r)} \right] f'(r) + V(\Phi) = 0, \quad (10)$$

where the prime and double prime are the first and the second derivatives with respect to r , respectively. Subtracting the equations (8) of (9) ($eq_{tt} - eq_{rr}$), we can obtain the following equation

$$\frac{R''(r)}{R(r)} + \frac{2R'(r)}{rR(r)} + 4\Phi'^2(r) = 0. \quad (11)$$

Next, we employ an ansatz, $R(r) = e^{2\alpha\Phi(r)}$, in the above field equation. The motivation for considering such an ansatz is due to black string solutions of Einstein-Maxwell-dilaton gravity which was first introduced in Ref. [113]. The $\Phi(r)$ is obtained

$$\Phi(r) = \frac{\gamma}{2} \ln \left(\frac{b}{r} \right), \quad (12)$$

where b is an arbitrary constant and $\gamma = \alpha^2 / K_{1,1}$ ($K_{i,j} = i + j\alpha^2$).

Here, in order to find consistent metric functions, we use a modified version of Liouville-type dilaton potential

$$V(\Phi) = 2\lambda e^{4\beta\Phi} + 2\Lambda e^{4\alpha\Phi}, \quad (13)$$

where Λ is a free parameter which plays the role of cosmological constant, λ and β are arbitrary constants. This potential is the usual Liouville-type dilaton potential that is used in the context of Friedman-Robertson-Walker (FRW) scalar field cosmologies [109] and Einstein-Maxwell-dilaton black holes [110–112]. Substituting Eqs. (12) and (13) in (7) and using the ansatz, $R(r) = e^{2\alpha\Phi(r)}$ in the field equation (10), we can construct exact charged black hole solutions of this theory by finding the metric functions, $f(r)$, as

$$f(r) = \frac{2\mathcal{K}_{1,1}^2 \Lambda r^2}{\mathcal{K}_{-2,1}} \left(\frac{b}{r} \right)^{2\gamma} - m r^\gamma + \frac{(2q^2)^s (2s-1)^2 \mathcal{K}_{1,1}^2}{\mathcal{K}_{2,1} - 2s} r^{\frac{2(s-1)}{2s-1}} \left[1 + \frac{\alpha^2 (s-1)}{(\alpha^2 - s\mathcal{K}_{-1,1})} \right], \quad (14)$$

where m is an integration constant which is related to the total mass of black holes. It is notable that, the above solutions (Eqs. (12) and (14)) will fully satisfy the system of equations provided,

$$\lambda = \frac{\alpha^2 (s-1) (2s-1) 2^{s-1} q^{2s}}{(\alpha^2 + s\mathcal{K}_{1,-1}) b^{\frac{2s}{2s-1}}}, \quad (15)$$

$$\beta = \frac{s\alpha}{\gamma(2s-1)}. \quad (16)$$

On the other hand, one can easily show that the vector potential A_μ , corresponding to the electromagnetic tensor (7), can be written as

$$A_\mu = \frac{(2s-1)qb^\gamma\mathcal{K}_{1,1}}{2s-\mathcal{K}_{2,1}} r^{\frac{2s-\mathcal{K}_{2,1}}{(2s-1)\mathcal{K}_{1,1}}}, \quad (17)$$

The electromagnetic gauge potential should be finite at infinity, therefore, one should impose following restriction to have this property

$$\frac{2s-\mathcal{K}_{2,1}}{(2s-1)\mathcal{K}_{1,1}} < 0. \quad (18)$$

The above equation leads to the following restriction on the range of s

$$\frac{1}{2} < s < \frac{2+\alpha^2}{2}. \quad (19)$$

It is notable that, in the absence of a non-trivial dilaton ($\alpha = \gamma = 0$), the solutions (14) reduce to

$$\Psi(r) = -m - \Lambda r^2 + \frac{(2q^2)^s (2s-1)^2}{2(1-s)} r^{\frac{2(s-1)}{2s-1}}. \quad (20)$$

which describes a 3-dimensional charged black hole in Einstein-PMI gravity, provided the PMI parameter must be in the range $\frac{1}{2} < s < 1$. It is notable that, for $s = \frac{3}{4}$ ($s = \frac{d}{4}$, in which d denote dimensions), this solution (Eq. (20)) reduce to conformally invariant Maxwell black hole solutions, as expected [114].

In order to confirm the black hole interpretation of the solutions, we look for the curvature singularity. To do so, we calculate the Kretschmann scalar. Calculations show that for finite values of radial coordinate, the Kretschmann scalar is finite. On the other hand, for very small and very large values of r , we obtain

$$\lim_{r \rightarrow 0} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \infty, \quad (21)$$

$$\lim_{r \rightarrow \infty} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \frac{16\Lambda^2(2\alpha^4 - 4\alpha^2 + 3)}{\mathcal{K}_{-2,1}^2} \left(\frac{b}{r}\right)^{4\gamma}. \quad (22)$$

The equation (21) confirms that there is an essential singularity located at $r = 0$, while Eq. (22) shows that for nonzero α , the asymptotical behavior of solutions is not (A)dS. It is easy to show that the metric function may contain real positive roots (see Fig. 1), and therefore, the curvature singularity can be covered with an event horizon and interpreted as a black hole.

III. THERMODYNAMICAL QUANTITIES

Now, we are in a position to calculate thermodynamic and conserved quantities of the obtained solutions and examine the validity of first law of thermodynamics.

In order to study the temperature, we use the concept of surface gravity which leads to

$$T = -\frac{\Lambda\mathcal{K}_{1,1}}{2\pi} b^{2\gamma} r_+^{\frac{\mathcal{K}_{1,-1}}{\mathcal{K}_{1,1}}} + \frac{2^{s-1}s(s-\frac{1}{2})q^{2s}\mathcal{K}_{1,1}}{\pi(s\mathcal{K}_{-1,1}-\alpha^2)r_+^{1/(2s-1)}}. \quad (23)$$

On the other hand, one can use the area law for extracting modified version of the entropy related to the Einsteinian class of black objects with the following structure

$$S = \frac{\pi r_+}{2} \left(\frac{b}{r_+}\right)^\gamma, \quad (24)$$

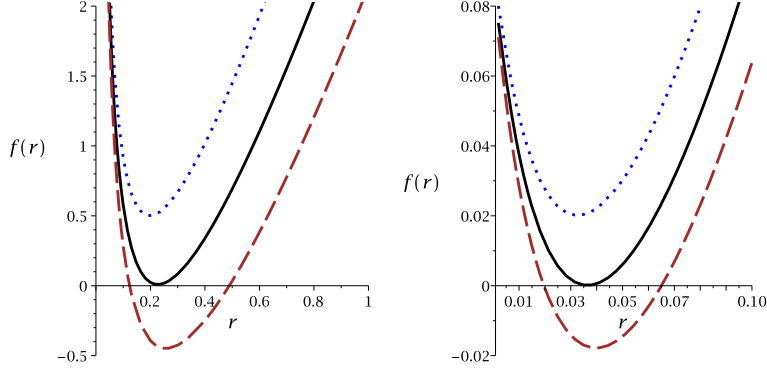


FIG. 1: $f(r)$ versus r for $\Lambda = -1$, $q = 0.3$, $\alpha = 1$ and $b = 1$.

left panel: for $s = 0.75$, $m = 4.0$ (dotted lines), $m = 5.1$ (continuous line) and $m = 6.0$ (dashed line).

right panel: for $s = 1.25$, $m = 6.0$ (dotted lines), $m = 6.1$ (continuous line) and $m = 6.2$ (dashed line).

in which by setting $\alpha = 0$, the entropy of charged BTZ black holes is obtained. In order to find the total electric charge of solutions, one can use the Gauss law. Calculating the flux of electric field helps us to find the total electric charge with the following form

$$Q = \frac{q^{2s-1} 2^{s-2} s^2}{s\mathcal{K}_{-1,1} + \alpha^2}. \quad (25)$$

Next, we are interested in obtaining the electric potential. Using following standard relation, one can obtain the electric potential at the event horizon with respect to the infinity as a reference

$$U(r) = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r \rightarrow r_+} = \frac{(2s-1)qb^\gamma \mathcal{K}_{1,1}^{\frac{2s-\mathcal{K}_{2,1}}{(2s-1)\mathcal{K}_{1,1}}}}{\mathcal{K}_{2,1} - 2s} r_+^{\frac{2s-\mathcal{K}_{2,1}}{(2s-1)\mathcal{K}_{1,1}}}. \quad (26)$$

Finally, according to the definition of mass due to Abbott and Deser [115–117], the total mass of solution is

$$M = \frac{b^\gamma}{8\mathcal{K}_{1,1}} m. \quad (27)$$

It is worthwhile to mention that for limiting case of $\alpha = 0$, Eq. (27) reduces to the mass of charged BTZ black holes [112].

Now, we are in a position to check the validity of first law of thermodynamics. To do so, first, we calculate the geometrical mass, m , by using $f(r = r_+) = 0$. Then by employing the obtained geometrical mass, we rewrite the mass relation, Eq. (27), in the following form

$$M(r_+, q) = \frac{b^{3\gamma} \mathcal{K}_{1,1} r_+^{\frac{\mathcal{K}_{2,-1}}{\mathcal{K}_{-2,1}}}}{4\mathcal{K}_{-2,1}} \Lambda + \frac{s(2s-1)^2 (2q^2)^s b^\gamma \mathcal{K}_{1,1} r_+^{\frac{2s-\mathcal{K}_{2,1}}{\mathcal{K}_{1,1}(2s-1)}}}{8(s\mathcal{K}_{-1,1} - \alpha^2)(2s - \mathcal{K}_{2,1})}. \quad (28)$$

It is a matter of calculation to show that

$$\left(\frac{\partial M}{\partial S}\right)_Q = T \quad \& \quad \left(\frac{\partial M}{\partial Q}\right)_S = U. \quad (29)$$

Therefore, it is proved that the first law is valid as

$$dM = \left(\frac{\partial M}{\partial S}\right)_Q dS + \left(\frac{\partial M}{\partial Q}\right)_S dQ. \quad (30)$$

IV. THERMODYNAMICAL BEHAVIOR

A. Mass

In usual black holes thermodynamics, the mass of black hole is interpreted as internal energy. In classical black hole thermodynamics, it is necessary to have positive value for internal energy of the system, hence, the mass of black

holes. Therefore, in this section we will study the conditions which are determining the positivity and negativity of this conserved quantity.

The mass of these black holes (28) is constructed by two terms. The first term of Eq. (28) is related to dilatonic gravity, and the second term includes a combination of electromagnetic and dilatonic parameters. The positivity of first term is determined by $K_{-2,1} > 0$. The second term has two specific parts which determine the positivity and negativity of it. These terms are $(s\mathcal{K}_{-1,1} - \alpha^2)$ and $(2s - \mathcal{K}_{2,1})$. In order for the second term has positive effects on the total mass of black holes, the following conditions should be satisfied simultaneously

$$\begin{aligned} (s\mathcal{K}_{-1,1} - \alpha^2) &> 0 \quad \& \quad (2s - \mathcal{K}_{2,1}) > 0, \\ (s\mathcal{K}_{-1,1} - \alpha^2) &< 0 \quad \& \quad (2s - \mathcal{K}_{2,1}) < 0. \end{aligned}$$

The existence of mentioned conditions for positivity and negativity of the mass indicates that under certain circumstances (suitable choices of different parameters), it is possible for the mass of black holes to be; completely negative/positive or a root may exist for the mass of black holes which results into existence of two regions of negative and positive mass for black holes. We see that the positivity/negativity of first term of Eq. (28) only depends on dilatonic gravity while for the second term of Eq. (28), due to the coupling of dilatonic gravity and electromagnetic field in the action, it depends on nonlinearity and dilatonic parameters.

B. Temperature

Now, we focus on the behavior of temperature. Thermodynamically speaking, for black holes, existence of negative temperature indicates non-physical systems, while, positive temperature is denoted as systems being physical. In other words, negativity and positivity of the temperature confirm whether obtained solutions for black holes are physical or non-physical ones. Here, the temperature of these black holes (23) contains two terms; Λ term which is purely due to the dilatonic part of action and electric charge term which is a combination of dilatonic parameter and electric charge.

The total contribution of Λ term depends on the choices of Λ itself. If $\Lambda > 0$, the contribution of this term will be towards the negativity whereas for $\Lambda < 0$, this term will have positive effects on values of temperature. As for the charge term, its effects are determined by two factors which are $(s - \frac{1}{2})$ and $(s\mathcal{K}_{-1,1} - \alpha^2)$. Considering our earlier discussions, $(s > \frac{1}{2})$, one can automatically conclude that the contributions of this term is positive. Therefore, the negativity or positivity of charge term only depends on the sign of $(s\mathcal{K}_{-1,1} - \alpha^2)$. Meaning that for

$$s\mathcal{K}_{-1,1} - \alpha^2 > 0,$$

the charge term will be positive and its effects are toward positivity of the temperature, whereas the opposite is observed for violation of this condition. The above condition presents the direct interaction of electromagnetic and dilatonic fields in an elegant way. We see that the coupling of electromagnetic field and dilaton gravity results into specific contribution in thermodynamical structure of the solutions and presents a tool for tuning the effects of dilaton gravity and electromagnetic field for specific purposes. Returning to temperature, interestingly, we notice that in Λ term, the power of event horizon only depends on dilatonic parameter whereas in charge term, it only depends on nonlinearity parameter, s . Here, we see that regardless of the coupling between electromagnetic field and dilatonic gravity, the power of horizon radius has been separated into two groups depending on different parts of the action. Thermodynamically speaking, the effectiveness of horizon radius has been separated into two branches which depends on the choices of dilatonic and nonlinearity parameters. The structure of action introduced such property into thermodynamical structure of the black holes. Therefore, it is possible to modify the effect of horizon radius in one branch (whether electromagnetic part or dilatonic gravity) without any concern regarding the other part.

C. Heat capacity

Now, we study the positivity/negativity of heat capacity. The heat capacity has specific information regarding the stability/instability of system and phase transitions. The phase transitions usually take place when system is in an unstable state. In other words, unstable systems go under a phase transition to acquire stability. The phase transition in heat capacity are recognized when the heat capacity meets a discontinuity, hence divergency. In other words, divergencies of the heat capacity are where systems go under phase transitions. The thermal stability/instability

of the system is determined by the sign of heat capacity; the negative values are representing the system being in unstable state while the positivity of the heat capacity is denoted as system being thermally stable.

The heat capacity is given by

$$C_Q = \left(\frac{\partial M}{\partial S} \right)_Q \left(\frac{\partial^2 M}{\partial S^2} \right)_Q^{-1}, \quad (31)$$

where by using the first law of black holes thermodynamics, one can rewrite it as

$$C_Q = T \frac{\left(\frac{\partial S}{\partial r_+} \right)_Q}{\left(\frac{\partial T}{\partial r_+} \right)_Q}. \quad (32)$$

Using this relation, it is possible to extract two thermodynamical important points; bound and phase transition points.

$$\left\{ \begin{array}{ll} T = 0 & \text{bounded point} \\ \left(\frac{\partial T}{\partial r_+} \right)_Q = 0 & \text{phase transition point} \end{array} \right. \quad (33)$$

The bound point is where the heat capacity, hence temperature, meets a root. Since this point separates physical (positive temperature) from non-physical (negative temperature) solutions, it is called bound point. Using the obtained temperature (23) and entropy (24), it is a matter of calculation to show that the heat capacity for these black holes is obtained as

$$C_Q = \frac{\pi \left[\Lambda (\alpha^2 - s\mathcal{K}_{-1,1}) b^{2\gamma} r_+^{(2s\mathcal{K}_{3,1} + \mathcal{K}_{-2,1})} + s (2q^2)^s \left(s - \frac{1}{2} \right) r_+^{(4s\mathcal{K}_{1,1} - \mathcal{K}_{2,1})} \right]}{2r_+ \left[\Lambda \mathcal{K}_{-1,1} (s\mathcal{K}_{-1,1} - \alpha^2) b^\gamma r_+^{2(s+\alpha^2)} - \frac{s(2q^2)^s \mathcal{K}_{1,1}}{2b^\gamma} r_+^{2s\alpha^2} \right]}. \quad (34)$$

The numerator and denominator of heat capacity consist of Λ , dilatonic parameter and electric charge. For having thermally stable solutions, hence positive heat capacity, two different set of conditions exist which must be satisfied. These conditions are

$$\Lambda (\alpha^2 - s\mathcal{K}_{-1,1}) > 0 \quad \& \quad \Lambda \mathcal{K}_{-1,1} (s\mathcal{K}_{-1,1} - \alpha^2) b^\gamma r_+^{2(s+\alpha^2)} - \frac{s(2q^2)^s \mathcal{K}_{1,1}}{2b^\gamma} r_+^{2s\alpha^2} > 0,$$

$$\frac{\Lambda (\alpha^2 - s\mathcal{K}_{-1,1}) b^{2\gamma}}{r_+^{-(2s\mathcal{K}_{3,1} + \mathcal{K}_{-2,1})}} + \frac{s(2q^2)^s \left(s - \frac{1}{2} \right)}{r_+^{(\mathcal{K}_{2,1} - 4s\mathcal{K}_{1,1})}} < 0 \quad \& \quad \frac{\Lambda \mathcal{K}_{-1,1} (s\mathcal{K}_{-1,1} - \alpha^2) b^\gamma}{r_+^{-2(s+\alpha^2)}} - \frac{s(2q^2)^s \mathcal{K}_{1,1}}{2b^\gamma r_+^{-2s\alpha^2}} < 0.$$

Respectively, the root (bound point) and divergence points (phase transition) of these black holes are given by

$$r_{root} = \left(\frac{\Lambda b^{2\gamma} (s\mathcal{K}_{-1,1} - \alpha^2)}{s(2q^2)^s \left(s - \frac{1}{2} \right)} \right)^{\frac{\mathcal{K}_{1,1}(2s-1)}{2(s\mathcal{K}_{-1,1} - \alpha^2)}}, \quad (35)$$

$$r_{phase \text{ transition}} = \left(\frac{2\Lambda b^{2\gamma} \mathcal{K}_{-1,1} (s\mathcal{K}_{-1,1} - \alpha^2)}{s(2q^2)^s \mathcal{K}_{1,1}} \right)^{\frac{\mathcal{K}_{1,1}(2s-1)}{2(s\mathcal{K}_{-1,1} - \alpha^2)}}. \quad (36)$$

If $\frac{\mathcal{K}_{1,1}(2s-1)}{2(s\mathcal{K}_{-1,1} - \alpha^2)} > 1$, then root and divergence points of the heat capacity are decreasing functions of the electric charge and increasing functions of Λ . If $\frac{\mathcal{K}_{1,1}(2s-1)}{2(s\mathcal{K}_{-1,1} - \alpha^2)}$ is even, then the existence of positive real valued root and phase transition point for these black holes is restricted to the following inequalities being satisfied

$$\Lambda (s\mathcal{K}_{-1,1} - \alpha^2) > 0, \quad \& \quad \Lambda \mathcal{K}_{-1,1} (s\mathcal{K}_{-1,1} - \alpha^2) > 0,$$

TABLE I: Roots of mass (r_M), temperature (r_T) and denominator of the heat capacity ($r(\frac{\partial T}{\partial r_+})$) for $q = 2$ and $b = 1$.

s	α	Λ	r_M	r_T	$r(\frac{\partial T}{\partial r_+})$
0.9	0	-1	<i>none</i>	1.528881	<i>none</i>
0.9	1	-1	<i>none</i>	1.973634	<i>none</i>
0.9	2	-1	0.559382	2.469012	7.636975
0.9	5	-1	0.121752	0.318611	0.805420
0.9	8	-1	<i>none</i>	0.017365	0.042973
0.7	1	-1	<i>none</i>	0.815298	<i>none</i>
0.8	1	-1	<i>none</i>	1.152417	<i>none</i>
1.1	1	-1	<i>none</i>	9.452169	<i>none</i>
1.2	1	-1	<i>none</i>	25.77402	<i>none</i>
1.3	1	-1	<i>none</i>	80.47861	<i>none</i>
0.9	1	0	<i>none</i>	<i>none</i>	<i>none</i>
0.9	1	1	1.426900	<i>none</i>	<i>none</i>

which once more, highlight the interaction between dilaton and electromagnetic fields.

In order to elaborate the effects of different parameters on thermodynamical behavior of the mass, temperature and heat capacity, we present a table (table I) and some diagrams (Figs. 2-5).

Evidently, depending on the choices of different parameters, these black holes may enjoy a phase transition in their phase diagrams. This phase transition is between large unstable black holes to small stable ones (see up panel of Fig. 2). The region of stability for this case is located between a bound and divergence point, whereas after the divergency, black holes are within physical region (positive temperature and mass) but are unstable (see up panel of Fig. 2). In another case, only a bound point exists where the physical stable solutions are observed after the bound point.

The critical horizon radius in which phase transition takes place is a decreasing function of the dilatonic parameter. Whereas, the bound point depending on the range of dilatonic parameter, could be increasing or decreasing function of dilatonic parameter (see table I and also Fig. 2). On the other hand, the bound point is an increasing function of the nonlinearity parameter (see Figs. 3 and 4). Also, there is one bound point for the negative values of cosmological constant and for other values of the cosmological constant (positive and zero) the black holes are not within physical region (see Fig. 5). By taking a closer look at the table I, one can notice that it is possible to obtain root for mass as well. This indicates that there exists a region where internal energy of the system is negative. This shows that the behavior of mass of the black hole imposes specific restrictions for solutions being physical/non-physical ones.

For the past few years, the studies that were conducted regarding thermodynamics of the black holes, have neglected the mass of black hole and its thermodynamical behavior. Here, we have shown that thermodynamical behavior of this quantity and its positivity/negativity play crucial roles for studying thermodynamics of the black holes. In fact, without having a well defined and positive internal energy (mass) for black holes and considering its behavior, assumptions and physical statements regarding to thermodynamical structure of the black holes will not be complete and even in some cases it could be wrong and misleading. The results that are derived for quantities such as temperature, heat capacity and etc. should be evaluated by considering the behavior of mass as well. Remembering that some important applications of the black holes and their thermodynamics, lie within AdS/CFT correspondence, the necessity of studying the mass of black holes in more details could be highlighted.

V. GEOMETRICAL THERMODYNAMICS

In this section, we employ geometrical thermodynamic approach to investigate thermodynamical properties of the black holes. The geometrical thermodynamic method provides the possibility of studying thermodynamical properties of the system through Riemannian geometry. In this method, one constructs thermodynamical phase space of the system and use its Ricci scalar to investigate thermodynamical properties. The divergencies of Ricci scalar in the phase space marks two important points: bound and phase transition points. In other words, divergencies (phase transition points) and bound points of the thermodynamical systems coincide with the divergencies of Ricci scalar. There are several approaches for constructing thermodynamical phase space which among them one can name; Weinhold

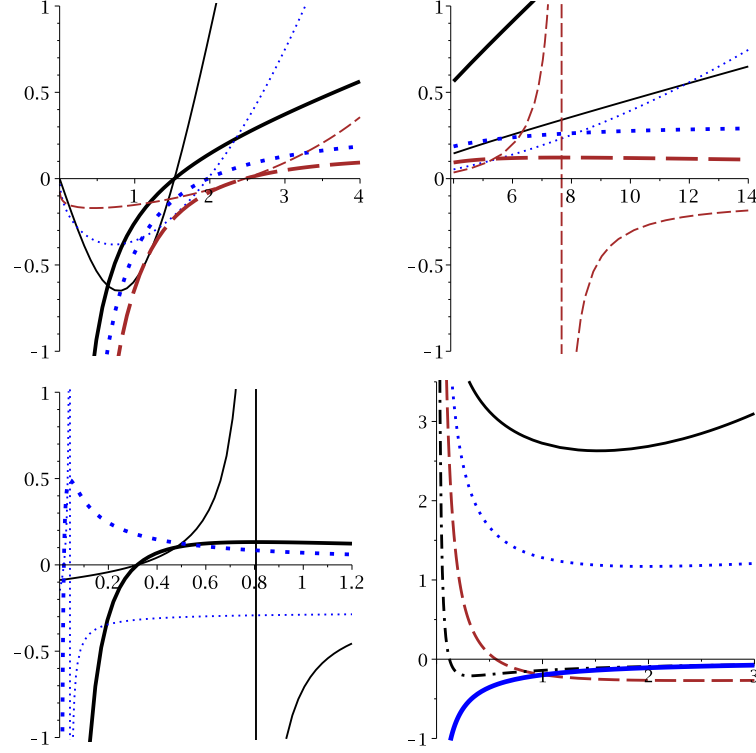


FIG. 2: For $\Lambda = -1$, $q = 2$, $s = 0.9$ and $b = 1$.

up panels: C_Q and T (bold lines) versus r_+ diagrams for $\alpha = 0$ (continuous line), $\alpha = 1$ (dotted line) and $\alpha = 2$ (dashed line).
down left panel: C_Q and T (bold lines) versus r_+ diagrams for $\alpha = 5$ (continuous line) and $\alpha = 8$ (dotted line).
down right panel: M versus r_+ diagrams for $\alpha = 0$ (continuous line), $\alpha = 1$ (dotted line), $\alpha = 2$ (dashed line), $\alpha = 5$ (dashed-dotted line) and $\alpha = 8$ (bold continuous line).

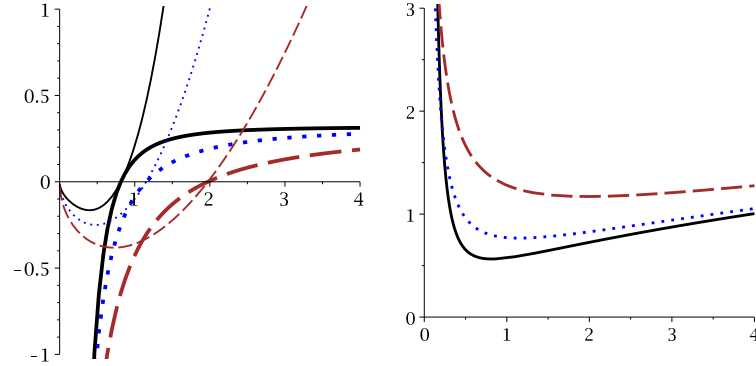


FIG. 3: For $\Lambda = -1$, $q = 2$, $\alpha = 1$ and $b = 1$; $s = 0.7$ (continuous line), $s = 0.8$ (dotted line) and $s = 0.9$ (dashed line)
left panel: C_Q and T (bold lines) versus r_+ diagrams.
right panel: M versus r_+ diagrams.

[89, 90], Ruppeiner [91, 92], Quevedo [93, 94] and HPEM [104–106]. Recent studies in the context of black holes thermodynamics revealed that Weinhold, Ruppeiner and Quevedo metrics may lead to inconsistent results regarding thermodynamical behavior of the system [104–106]. In other words, cases of mismatch between the divergencies of Ricci scalar and, bound and phase transition points or existence of extra divergencies unrelated to mentioned points were reported. To overcome the shortcomings of mentioned methods, HPEM formulation was introduced and it was shown that specific structure of this metric provides satisfactory results regarding the geometrical thermodynamics of different classes of black holes.

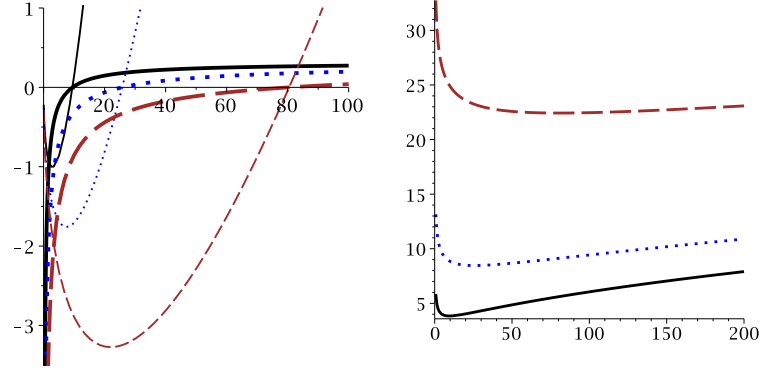


FIG. 4: For $\Lambda = -1$, $q = 2$, $\alpha = 1$ and $b = 1$; $s = 1.1$ (continuous line), $s = 1.2$ (dotted line) and $s = 1.3$ (dashed line)
left panel: C_Q and T (bold lines) versus r_+ diagrams.
right panel: M versus r_+ diagrams.

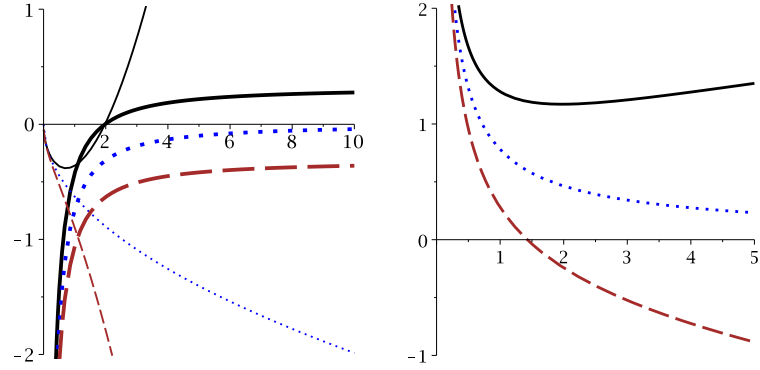


FIG. 5: For $s = 0.9$, $q = 2$, $\alpha = 1$ and $b = 1$; $\Lambda = -1$ (continuous line), $\Lambda = 0$ (dotted line) and $\Lambda = 1$ (dashed line)
left panel: C_Q and T (bold lines) versus r_+ diagrams.
right panel: M versus r_+ diagrams.

The mentioned thermodynamical metrics are in the following forms

$$ds^2 = \begin{cases} Mg_{ab}^W dX^a dX^b & \text{Weinhold} \\ -T^{-1} M g_{ab}^R dX^a dX^b & \text{Ruppeiner} \\ (SM_S + QM_Q) (-M_{SS} dS^2 + M_{QQ} dQ^2) & \text{Quevedo} \\ S \frac{M_S}{M_{QQ}^3} (-M_{SS} dS^2 + M_{QQ} dQ^2) & \text{HPM} \end{cases}, \quad (37)$$

where $M_X = \partial M / \partial X$ and $M_{XX} = \partial^2 M / \partial X^2$. It is a matter of calculation to show that the denominators of Ricci scalar of these phase spaces are [104]

$$\text{denom}(\mathcal{R}) = \begin{cases} M^2 (M_{SS} M_{QQ} - M_{SQ}^2)^2 & \text{Weinhold} \\ M^2 T (M_{SS} M_{QQ} - M_{SQ}^2)^2 & \text{Ruppeiner} \\ M_{SS}^2 M_{QQ}^2 (SM_S + QM_Q)^3 & \text{Quevedo} \\ 2S^3 M_{SS}^2 M_S^3 & \text{HPM} \end{cases}. \quad (38)$$

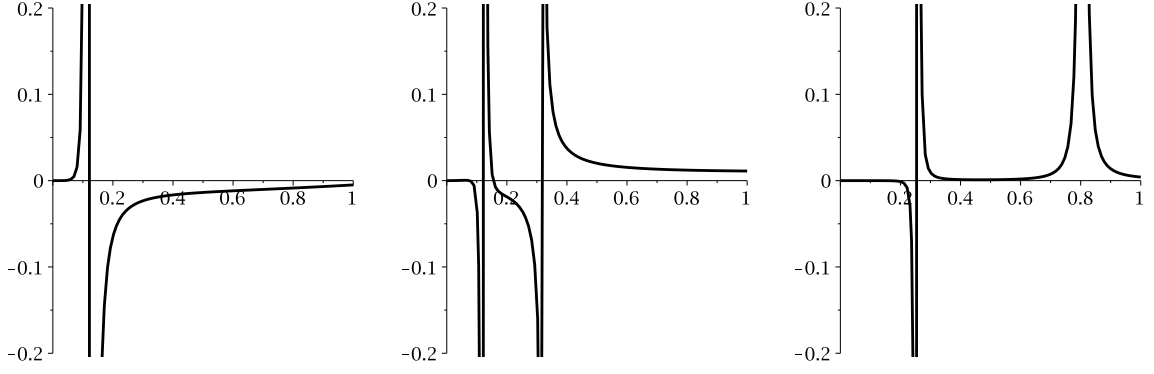


FIG. 6: \mathcal{R} versus r_+ for $q = 2$, $b = 1$, $\alpha = 5$, $s = 0.9$ and $\Lambda = -1$;
left panel: Weinhold; middle panel: Ruppeiner; right panel: Quevedo.

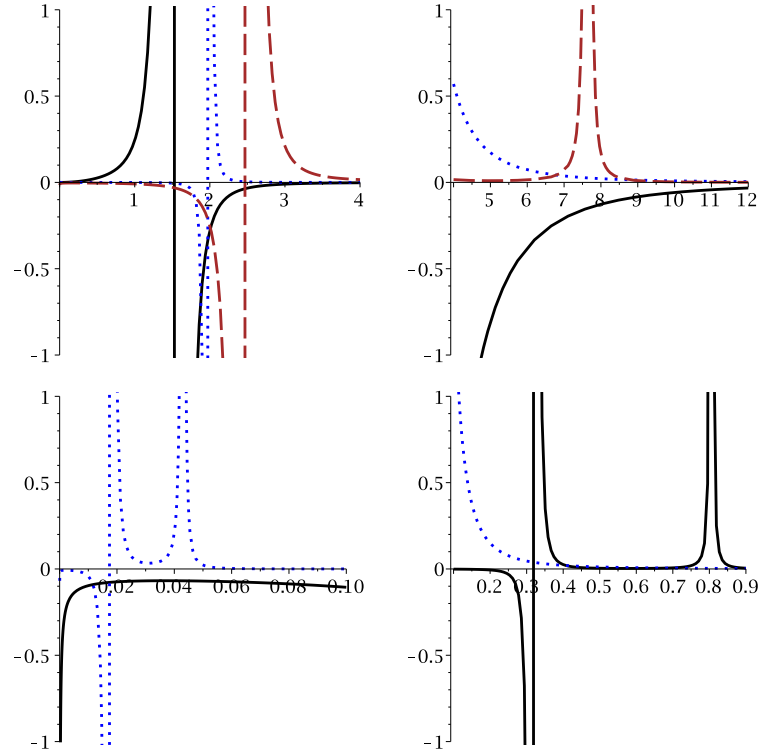


FIG. 7: For $\Lambda = -1$, $q = 2$, $s = 0.9$ and $b = 1$.
up panels: \mathcal{R} versus r_+ diagrams for $\alpha = 0$ (continuous line), $\alpha = 1$ (dotted line) and $\alpha = 2$ (dashed line).
down panels: $\alpha = 5$ (continuous line) and $\alpha = 8$ (dotted line).

In order to have consistent results regarding bound and phase transition points, the denominator of Ricci scalar of each metric must yield M_{SS} and M_S , explicitly. Here, we see that HPEM metric has such factors in its denominator of Ricci scalar. Whereas, the Quevedo metric has only M_{SS} in its Ricci scalar's denominator and Weinhold and Ruppeiner do not have these factors in explicit forms (see [104] for more details). In other words, the structures of denominators of the Ricci scalars obtained of Weinhold, Ruppeiner and Quevedo metrics may yield extra divergencies for their Ricci scalars which are not related to any bound and phase transition points. In addition, these Ricci scalars may admit a mismatch between bound and phase transition points and divergencies of the Ricci scalar. To elaborate such cases, we have plotted Fig. 6.

Evidently, for the Weinhold case, only one divergency for its Ricci scalar is observed which is not matched with any bound and phase transition points (compare left panel of Fig. 6 with down left panel of Fig. 2). The Ruppeiner metric has two divergencies for its Ricci scalar; one is matched with the bound point while the other one is not

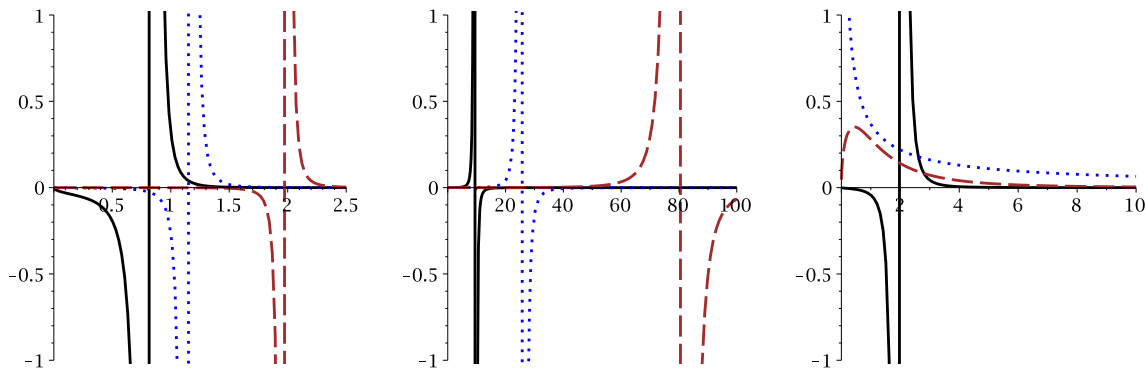


FIG. 8: \mathcal{R} versus r_+ for $q = 2$, $\alpha = 1$ and $b = 1$;
left panel: $\Lambda = -1$, $s = 0.7$ (continuous line), $s = 0.8$ (dotted line) and $s = 0.9$ (dashed line).
middle panel: $\Lambda = -1$, $s = 1.1$ (continuous line), $s = 1.2$ (dotted line) and $s = 1.3$ (dashed line).
right panel: $s = 0.9$, $\Lambda = -1$ (continuous line), $\Lambda = 0$ (dotted line) and $\Lambda = 1$ (dashed line).

matched with any phase transition point (compare middle panel of Fig. 6 with down left panel of Fig. 2). On the other hand, there are two divergencies for Quevedo metric. One of them is coincidence with the phase transition point while the other one is not matched with the bound point (compare right panel of Fig. 6 with down left panel of Fig. 2). It is evident that employing these three metrics leads into results which are not consistent regarding the bound and phase transition points of these black holes. In order to have consistent results, we employ the HPEM metric and plot following diagrams (Figs. 7 and 8).

A simple comparison between Figs. 7 and 8 with plotted diagrams of the previous section (Figs. 2 and 5) shows that all the bound and phase transition points are matched with divergencies of the Ricci scalar of HPEM metric for different cases. This confirms the validation of the results of HPEM metric. Therefore, one can employ this method as an independent approach for studying thermodynamical properties of the black holes. This is the main purpose of the geometrical thermodynamics. On the other hand, by taking a closer look, one can see that the sign of HPEM Ricci scalar around the bound and phase transition points, depends on the type of point. While we have a change of sign for Ricci scalar around the bound point, but for the phase transition point, the sign of HPEM Ricci scalar remains fixed. Therefore, it is possible to distinguish the type of point by studying the sign of Ricci scalar of this metric. In the geometrical thermodynamics, the sign of Ricci scalar determines whether the system has repulsive or attractive interaction around the bound and phase transition point. The positive sign indicates that system has repulsive interaction whereas the opposite could be said about negative sign. Here, we see that before the bound point, system has attractive property and by crossing the bound point, the interaction is changed into repulsive. On the other hand, for the phase transition point, the sign of Ricci scalar is positive. Therefore, here, system has repulsive interaction on fundamental level. Also, we see that employing the geometrical thermodynamics provides us with extra information regarding the nature of interactions around the bound and phase transition points. These information could not be extracted by using the temperature and heat capacity of a system.

VI. CONCLUSION

We studied three dimensional dilatonic black holes in the presence of a nonlinear electromagnetic field known as power Maxwell invariant. It was shown that the solutions have interpretation of the black holes. Conserved and thermodynamical quantities of these solutions were extracted and it was shown that the first law of black holes thermodynamics is satisfied.

In order to highlight the importance and the role of mass (internal energy), a thermodynamical investigation with the analyzing mass, temperature and heat capacity of the black holes was done.

It was shown that it is possible to obtain roots for the mass of black holes which led to the presence of region where the mass of black holes was negative. This shows that in order to have a better picture regarding thermodynamical structure of the black holes, it is necessary to include the behavior of mass as well. There are specific restrictions which are imposed by thermodynamical behavior of the mass which could not be neglected and must be taken into account for having reliable conclusions and predictions regarding thermodynamics of the black holes. For the past decades, the studies regarding thermodynamical structures of the black holes have neglected this important and crucial factor. Considering that we are using thermodynamics of the black holes in AdS spacetime for applications in conformal field

theory, the restrictions which are imposed by mass of the black holes become highly important.

In addition, the effects of different parameters on thermodynamical structures and phase transitions of these black holes were investigated. It was shown that the existence of phase transition is depending on the values of different parameters.

Next, geometrical thermodynamics was employed to study thermodynamical structure of these black holes. It was shown that for these specific black holes, using Weinhold, Ruppeiner and Quevedo methods lead to inconsistent results regarding the bound and phase transition points, whereas, HPEM approach was successful in describing thermodynamical properties of these black holes. Furthermore, we employed the sign of HPEM metric around the bound and phase transition point to study repulsiveness and attractiveness of the interactions.

Finally, it is worthwhile to generalize the obtained results to the case of non-abelian Yang-Mills field. In addition, it will be interesting to study causal structure of the solutions and examine the possibility of closed timelike curves in the special case of dilaton and nonlinearity parameters. Furthermore, one may pay attention to the possibility of having three dimensional hairy black holes in the context of massive gravity. We will address these subjects in the future works.

Acknowledgments

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